Parametric Constraints with Application to Optimization for Flutter Using a Continuous Flutter Constraint

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Many optimum structural-design problems involve constraints which have to be satisfied for an entire range of certain parameters. For example, in a structure under dynamic loads the stress constraint has to be satisfied for a given range of time. The properties of parametric constraints are investigated in the present paper. It is shown that a parametric constraint may be replaced by an equivalent "minimum value" constraint which is more efficient computationally for optimization problems. The efficiency of the minimum value constraint is due to the minimal amount of computation needed to update it as the structure is changed in the optimization process. The results obtained for the general case of parametric constraints are applied to the problem of optimal structural design under flutter constraints. It is shown that a recently developed continuous flutter constraint is a parametric constraint and may be replaced by the equivalent minimum value constraint. An example problem is solved using the continuous flutter constraint. It is shown that the design obtained for this problem could not be obtained by using a more conventional flutter constraint.

Nomenclature

A_{ij} = generalized aerodynamic forces
b_o = wing semiroot chord
C = flutter matrix, Eq. (19)
G = constraint function
G_f = continuous flutter constraint function, Eq. (22)
G_{min} = minimum value constraint function, Eq. (4)
g = artificial damping, Eq. (20)
g_j = structural damping of jth vibration mode
g_{ref} = reference structural damping
k = reduced frequency, $k = b_o \omega / V$
l = wing semispan
M = Mach number
M_j = generalized mass of jth vibration mode
n = number of vibration modes used in the flutter analysis
N = number of design variables
q_j = generalized coordinate, jth vibration mode
r = number of constraints
t = thickness
t_i = thickness of cover panels for ith segment on wing
V = air speed
V_f = flutter speed
V_{req} = required flutter speed
v = design variable vector
v_j = jth design variable
Λ = constraint parameter set
Λ_c = critical constraint parameter set
λ = constraint parameter
λ_m = location of minimum value constraint, Eq. (5)
ρ = air density
Ω_r = flutter eigenvalue, Eq. (20)
ω = flutter frequency
ω_j = frequency of jth vibration mode
ω_r = reference frequency

Introduction

OPTIMUM-structural design is concerned with finding a structure that satisfies safety and performance constraints at least costs. A typical constraint is the stress constraint, which specifies the maximum value of the stresses at any point of the structure. In a case of a structure under dynamic loading, the stresses have to remain below their allowable values at all times. In this case the stress constraint is a parametric constraint, time being the parameter. In general, constraints which have to be satisfied for an entire range of certain parameters are called here parametric constraints.

The dependence of the constraint on the parameter may be simple enough so that it is possible to check whether the constraint is violated somewhere in the parameter range. In many cases, however, the dependence is complex and the constraint has to be checked for a large number of parameter values to ensure that it is not violated, and this may prove to be computationally expensive.

In an optimization problem many configurations of the structure are examined, corresponding to different combinations of design variables which determine the shape, size, or material of structural elements. For each configuration the constraints have to be checked for the entire range of their parameters. In the example of a structure under dynamic loading the stresses have to be computed at all times for each new configuration. This, of course, magnifies the computational problem.

One way of reducing the computational effort is to use information gained when a constraint is first evaluated to reduce the set of parameter values at which it is enforced for subsequent steps of the optimization process. An obvious way of achieving this is to discard parameter values for which the constraint is satisfied with a wide margin (in the example above, discard time intervals where stresses are small compared to their allowable values). This is the "posture table" approach of Schmit and Farshi.[1] The posture table is a list of values for which the constraint is nearly critical, the margin being less than a specified small value. The posture table is updated periodically during the optimization process by recalculating the constraint for the entire parameter range, but between updatings the constraint is calculated only for the parameter values in the posture table.

The present paper attempts a different approach to the problem of reducing the computational effort of repeated calculation of a parametric constraint. The approach is based on replacing the parametric constraint by an equivalent nonparametric "minimum value" constraint. The equivalent "minimum-value" constraint is essentially the requirement that the minimum value of the margin of the original parametric constraint over the

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parameter range be positive. As for the posture table approach, the entire range of the parameter has to be checked the first time the constraint is evaluated in order to find the minimum value. However, for subsequent steps in the optimization process it is shown that very few parameter values are needed to update the constraint. (These are the parameter values for which the minimum is attained). It may be expected that much fewer parameter values would be needed than by using the posture table approach.

The results obtained for the general case of parametric constraints are applied to the problem of optimal structural design under flutter constraints. One of the basic problems of design under flutter constraints is the discontinuous behaviour of some of the flutter parameters. For example, the flutter speed may be a discontinuous function of air density,[2] or structural stiffness.[3] Thus, if the flutter constraint is formulated as a limit on the flutter speed (as is usually done) the constraint may not be a continuous function of the structural design variables. This, in turn, may lead to a breakdown of an optimization procedure which requires continuity of constraint functions.

McCullers and Lynch proposed a flutter constraint based on an entire V-g diagram.[4] Their flutter constraint is continuous and it is a parametric constraint, the reduced frequency being the parameter. The continuous flutter constraint as formulated in Ref. 4 is computationally expensive. It is shown that by replacing it with an equivalent "minimum value" constraint the computational penalty involved with using a continuous flutter constraint may be greatly reduced. An example problem is solved using the continuous flutter constraint. It is shown that the design obtained for this problem could not be obtained by using a more conventional flutter constraint.

II. Parametric Constraints

Many problems of structural optimization may be posed as follows: minimize $F(v)$ subject to the constraints $G_i(v) > 0$, $i = 1,\ldots,r$ where v is some design variable vector (having the components $v_j, j = 1,\ldots, N$). This form, however, is suitable only for a discrete set of constraints. It often happens that we have a continuous set of constraints

$$G(\lambda, v) > 0 \qquad \lambda \in \Lambda \tag{1}$$

where λ is a parameter having some value in a specified set Λ.

One way (see Ref. 5, for example) of dealing with a continuous constraint is to integrate it over the set Λ, that is define

$$\bar{G}(v) = \{\textstyle\int_\Lambda [d\Lambda/G(\lambda, v)]\}^{-1} \tag{2}$$

$\bar{G}(v)$, the integrated constraint is positive when $G(\lambda, v)$ is positive for all λ in Λ. Also, $\bar{G}(v) \to 0$ when $G(\lambda, v) \to 0$ at any point λ. Another way to deal with a parametric constraint is to discretize it by enforcing it at a finite number of λ-values, that is, require

$$G(\lambda, v) > 0 \qquad \lambda = \lambda_I \ldots, \lambda_m \tag{3}$$

This way is natural when the constraint set has to be discretized for other reasons as well. For example, a stress constraint may require that the stress at any point of a structure is less than the yield stress. If the structure is discretized into finite elements then the stress constraint is often enforced at one point in each element.

Both ways of treating a parametric constraint have one serious drawback, they require the calculation of the constraint function $G(\lambda, v)$ at a large number of λ-values, which in the case of some constraints may be expensive. One way of reducing the amount of computation is to form a posture table of active constraints (see Ref. 1, for example) including only the set Λ_c such that

$$\lambda \in \Lambda_c \quad \text{if} \quad G(\lambda, v) < \delta$$

where δ is some suitably chosen small quantity. The optimization process is based on the critical set Λ_c with periodic updating of the set. The main disadvantages of this method are that 1) it is not always clear how to choose δ and 2) the set Λ_c may still be very large.

To remedy the above problem the following constraint is proposed

$$G_{\min}(v) = \min_\lambda G(\lambda, v) \qquad \lambda \in \Lambda \tag{4}$$

The "minimum value" constraint obviously is an appropriate constraint, that is $G_{\min}(v) > 0$ means that Eq. (1) is satisfied. The minimum is reached at a value of λ, λ_m, which is, of course, a function of v

$$G_{\min}(v) \equiv G[\lambda_m(v), v] \tag{5}$$

The minimum defined in Eq. (4) should be interpreted as a local minimum rather than a global one. Thus, if there are several local minima each would have a separate constraint associated with it. The reason for using local minima rather than the global minimum is that $\lambda_m(v)$ may be a discontinuous function of v if it describes the location of the global minimum, even if $G(\lambda, v)$ is a continuous and differentiable function. In the following only one $G_{\min}(v)$ is considered, but it is understood that it is one of possibly several minima. The usefulness of the minimum value constraint is that once it is evaluated for a point v in the design space it is easy to re-evaluate it at neighboring points. This is demonstrated now.

It is assumed that $G(\lambda, v)$ is a twice differentiable function of λ and v. A minimum of $G(\lambda, v)$ with respect to λ implies that the derivative of G vanishes so that

$$[\partial G(\lambda_m, v)/\partial\lambda] = 0 \tag{6}$$

Differentiating Eq. (5) with respect to a design variable v_i one obtains

$$\frac{\partial G_{\min}}{\partial v_i} = \frac{\partial G(\lambda_m, v)}{\partial v_i} + \frac{\partial G(\lambda_m, v)}{\partial \lambda}\frac{\partial \lambda_m}{\partial v_i} = \frac{\partial G(\lambda_m, v)}{\partial v_i} \tag{7}$$

Equation (7) means that for a small move Δv_i in the design space

$$G_{\min}(v+\Delta v) \cong G_{\min}(v) + \sum_{i=1}^{N} \Delta v_i \frac{\partial G_{\min}(v)}{\partial v_i} = G(\lambda_m, v) + \sum_{i=1}^{N} \Delta v_i \frac{\partial G(\lambda_m, v)}{\partial v_i} \cong G(\lambda_m, v+\Delta v) \tag{8}$$

That is, for a small move the minimum value constraint may be calculated as if λ_m is constant. Thus, for a small move, instead of recalculating $G(\lambda, v)$ for all λ's it is necessary to recalculate it only for one λ.

For larger moves one may want to use more terms in the Taylor expansion

$$G_{\min}(v+\Delta v) \cong G_{\min}(v) + \sum_{i=1}^{N} \Delta v_i \frac{\partial G_{\min}(v)}{\partial v_i} + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\Delta v_i \Delta v_j}{2} \frac{\partial^2 G_{\min}(v)}{\partial v_i \partial v_j} \tag{9}$$

where $\partial^2 G_{\min}/\partial v_i \partial v_j$ is obtained by differentiating Eq. (7)

$$\frac{\partial^2 G_{\min}}{\partial v_i \partial v_j} = \frac{\partial^2 G(\lambda_m, v)}{\partial v_i \partial v_j} + \frac{\partial^2 G(\lambda_m, v)}{\partial \lambda \partial v_i}\frac{\partial \lambda_m}{\partial v_j} \tag{10}$$

The derivative $\partial\lambda_m/\partial v_j$ is obtained by differentiating Eq. (6)

$$\frac{\partial^2 G(\lambda_m, v)}{\partial v_i \partial \lambda} + \frac{\partial^2 G(\lambda_m, v)}{\partial \lambda^2}\frac{\partial \lambda_m}{\partial v_i} = 0 \tag{11}$$

or

$$\frac{\partial \lambda_m}{\partial v_i} = -\frac{\partial^2 G(\lambda_m, v)}{\partial v_i \partial \lambda} \Bigg/ \frac{\partial^2 G(\lambda_m, v)}{\partial \lambda^2} \tag{12}$$

Substituting from Eqs. (5), (7), (10) and (11) into Eq. (9) one obtains

$$G_{\min}(v+\Delta v) \cong G(\lambda_m, v) + \sum_{i=1}^{N} \Delta v_i \frac{\partial G(\lambda_m, v)}{\partial v_i} + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\Delta v_i \Delta v_j}{2}\frac{\partial^2 G(\lambda_m, v)}{\partial v_i \partial v_j} - \frac{1}{2}\frac{\partial^2 G(\lambda_m, v)}{\partial \lambda^2}(\Delta\lambda_m)^2 \tag{13}$$

where $\Delta\lambda_m$ is

$$\Delta\lambda_m = \sum_{i=1}^{N} (\partial\lambda_m/\partial v_i)\Delta v_i \tag{14}$$

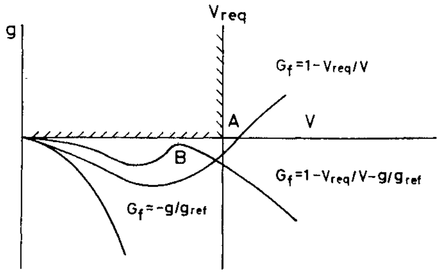

Fig. 1 A typical V-g diagram and the definition of the continuous constraint function G_f.

Equation (13) may also be written as

$$G_{\min}(v+\Delta v) = G(\lambda_m, v+\Delta v) - \frac{1}{2}\frac{\partial^2 G(\lambda_m, v)}{\partial \lambda^2}(\Delta\lambda_m)^2 \qquad (15)$$

Equation (15) gives the correction to the minimum value constraint due to the change in λ_m, and it is, indeed, a second-order effect.

III. Structural Design Under Flutter Constraints

The results obtained in Sec. II for the general case of parametric constraints are now applied to the problem of structural design under flutter constraints. The problem is to minimize the mass m of a wing subject to a number of flutter constraints; the typical constraint is usually formulated as a lower limit V_{req} on the flutter speed V_f at a given altitude, that is

$$1-(V_{\text{req}}/V_f) > 0 \qquad (16)$$

Design variables may be thicknesses of structural elements, magnitude of balancing masses, or any other geometrical or structural parameters.

As was noted in the introduction, the flutter speed may be a discontinuous function of the design variables. Therefore, another form of the constraint was proposed[4] which is a continuous function of design variables. The continuous flutter constraint is based on the V-g method for the solution of the flutter problem. The method is described here in brief.

V-g Method

The flutter equations are written for a wing having a root chord of $2b_o$ and a span $2l$. A small number, n, of the natural vibration modes are used as generalized coordinates. The modes have frequencies ω_j, generalized mass M_j and a structural damping coefficient g_j. The flutter equations [Ref. 6, Eq. (35)] are

$$\sum_{j=1}^{N}\left\{A_{ij}+[1-(\omega_j^2/\omega_r^2)(1+g_j)\Omega_r]\frac{k^2 M_j}{(\rho l b_o^2/2)}\delta_{ij}\right\}q_j = 0$$
$$i = 1,\ldots,n \qquad (17)$$

where A_{ij} = generalized aerodynamic forces, q_j = amplitude of jth mode, k = reduced frequency, $(b_o\omega/V)$, ω_r = reference frequency, ω = flutter frequency, V = flutter speed, ρ = air density, Ω_r = eigenvalue, (ω_r^2/ω^2). Equation (17) is an eigenvalue problem in Ω_r and may be written as

$$[C-\Omega_r I]\{q\} = 0 \qquad (18)$$

where I is the unit matrix and the matrix C is defined by

$$C_{ij} = \frac{(\rho l b_o^2/2)A_{ij}+k^2 M_j \delta_{ij}}{(\omega_j^2/\omega_r^2)(1+g_j)M_j} \qquad (19)$$

The generalized aerodynamic forces are a function of the density ρ, the Mach number M and the reduced frequency k. For certain combinations of ρ, k, and M, Ω_r is real which signifies a flutter point. The V-g method for finding the flutter point is based on prescribing the altitude (and thus ρ) and the Mach number and solving Eq. (18) for a series of k-values. To avoid the use of complex frequencies, Ω_r is redefined as

$$\Omega_r = (\omega_r^2/\omega^2)(1+ig) \qquad (20)$$

At a flutter point $g = 0$. For each value of k, ω and g are calculated from Eq. (20) and V is calculated from k. V and g are then plotted with k as a parameter as shown in Fig. 1. Because there are n eigenvalues of Eq. (18) for each value of k, n such curves are obtained. A flutter point corresponds to a crossing ($g = 0$) such as point A in Fig. 1.

Point B in Fig. 1 is not a crossing and therefore it does not represent a flutter point. It is, however, possible that a change in a structural parameter may cause the peak at point B to shift upwards and cross the $g = 0$ line. This type of flutter mode is referred to as a hump mode.

The surfacing of a hump mode above the $g = 0$ line may cause a discontinuity in the lowest flutter speed V_f and therefore in the constraint equation [Eq. (16)]. This discontinuity is not a physical one in the sense that the transition from a small value of positive damping (negative g) to a small value of negative damping (positive g) is continuous. It is possible therefore to reformulate the constraint equation so that it becomes continuous.

Continuous Flutter Constraint

The continuous flutter constraint[4] is based on a V-g diagram for a specified altitude and Mach number. Instead of Eq. (16) the flutter constraint is posed as

$$G_f(k) > 0 \qquad k_{\min} \leqq k \leqq k_{\max} \qquad (21)$$

where G_f is defined as shown in Fig. 1

$$G_f = 1-V_{\text{req}}/V_f \quad \text{if} \quad V_f > V_{\text{req}} \quad \text{and} \quad g > 0$$
$$G_f = -g/g_{\text{ref}} \quad \text{if} \quad V_f < V_{\text{req}} \qquad (22)$$
$$G_f = 1-V_{\text{req}}/V_f - g/g_{\text{ref}} \quad \text{if} \quad V_f > V_{\text{req}} \quad \text{and} \quad g < 0$$

$k_{\min}$, $k_{\max}$ define the range of frequencies which is of interest, and g_{ref} is a reference damping value.

The constraint as defined by Eq. (21) may include a safety factor. A safety factor on the flutter speed is usually included by increasing V_{req} by, say, 20% above the design speed (or, equivalently, increasing the dynamic pressure). For the continuous flutter constraint a safety factor is required also for the damping. If no safety factor is included a small change n the structure may cause a "hump mode" (such as point B in Fig. 1) to cross the $g = 0$ line and cause flutter at speeds below $V = V_{\text{req}}$. Herein it is assumed that a safety factor is built in by reducing the structural damping g_i used in the calculations.

The flutter constraint defined by Eq. (21) is a parametric constraint, k being the parameter. This constraint is replaced by an equivalent constraint

$$\min_k G_f(k) > 0 \qquad (23)$$

The location k_m of the minima is found for the initial design by tracing the entire V-g diagram.† Thereafter, one may use either the first-order or second-order approximations [Eqs. (8) or (15), respectively] and recalculate the entire V-g diagram only a few times during the design process. Further comments on the use of Eq. (15) for the continuous flutter constraint are given in the Appendix.

Computer Program

The WIDOWAC computer program[7] was used for the implementation of the continuous flutter constraint and later the "minimum value" version of this constraint. WIDOWAC was developed to size minimum-mass, symmetric-airfoil structures that satisfy flutter, strength, and minimum gage requirements. In WIDOWAC, a wing structure is modelled by a finite-element representation. Second-order piston theory aerodynamics is used for supersonic conditions and kernel function aerodynamics for subsonic aerodynamics. The program uses the mathematical

† It should be noted that not all the minima of $G_f(k)$ are obvious from a visual examination of a V-g diagram such as Fig. 1. Thus, while point B in Fig. 1 is an obvious local minimum of $G_f(k)$, it is also true that there is also a local minimum near the crossing point A in Fig. 1.

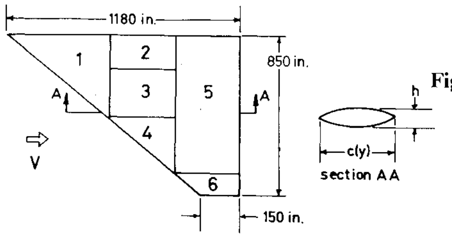

Fig. 2 Wing geometry.

programing approach of sequential unconstrained minimization (SUMT)[8] in which the design constraints are introduced by means of an interior penalty function. Newton's method with approximate second derivatives[3] is the optimization algorithm used for each unconstrained minimization.

The implementation of the original[4] continuous flutter constraint in WIDOWAC involved a penalty of 100% in computer time due to the need to calculate the entire V-g diagram for every change in the design variables. The "minimum value" equivalent formulation was then implemented but only for the calculation of derivatives [Eq. (7)]. That is, the V-g diagram is still calculated in its entirety for each step in a one dimensional search. Even this limited use of the "minimum value" approach resulted in reducing the penalty for the use of the continuous constraint to only about 30–40% of the computer time required for the standard flutter constraint [i.e., that of Eq. (16)].

Beryllium Patch on a Titanium Wing

An example of discontinuous flutter behavior given in Ref. 3 is that of a titanium wing with a beryllium patch (Fig. 2). The wing has a honeycomb sandwich core and cover panels which are made of titanium (segments 1, 2, 4, 5, 6 in Fig. 2) except for a beryllium patch (segment 3, Fig. 2). The flutter speed at an altitude of 25,000 ft is discontinuous when the thickness of the titanium cover panels is 0.1 in. while the thickness of the beryllium patch is increased above 0.15 in. The wing was analyzed by the WIDOWAC program[7] using a 93 degrees of freedom finite-element model (the model is given in Ref. 9).

Figures 3 and 4 give the V-g diagrams for the wing when $t_3 = 0.14$ in. and 0.16 in. respectively. The discontinuity is obviously caused by a "hump" mode crossing the $g = 0$ line. It should be noted that the two flutter modes discussed in Ref. 3 correspond to the hump mode and to the crossing at point B (Fig. 3 or Fig. 4). The wing was also optimized for minimum mass using continuous flutter constraint. Five design variables were used—the thicknesses of segments 1–5. The thickness of segment 6 was set to the minimum gage of 0.02 in. The constraints applied to the problem are: 1) Maximum stress of 125 ksi in titanium and 50 ksi in beryllium for a uniform loading of 1 psi. 2) Flutter speed not less than 2500 fps at 25,000 ft, assuming *zero structural damping*. 3) Minimum skin thickness, 0.02 in.

Segment thicknesses and the V-g diagram for the final design

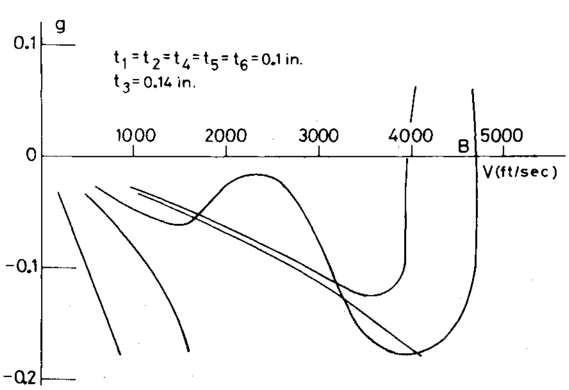

Fig. 3 V-g diagram, beryllium patch thickness 0.14 in.

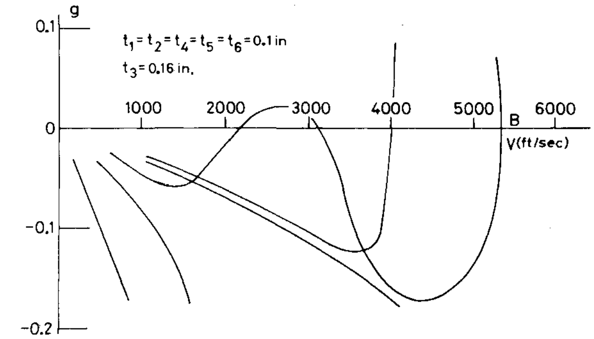

Fig. 4 V-g diagram, beryllium patch thickness 0.16 in.

are shown in Fig. 5. It is interesting to note that the lowest crossing is at a speed of 2830 fps well above the required speed of 2500 fps. The critical condition‡ is a hump mode with a peak at a speed of about 1850 fps. Therefore, an optimization method which is based on the assumption that for the optimum design the flutter speed is equal to the required speed cannot arrive at this design.

Conclusions

It was shown that a parametric constraint may be replaced by an equivalent nonparametric "minimum value" constraint. The main advantage of the minimum value constraint is that once it is evaluated for a point in the design space it is easy to re-evaluate it at neighboring points. It is therefore highly suitable for use in an optimization process.

The results obtained for the general case of parametric constraints were applied to the problem of optimal structural design under flutter constraints. It was shown that a recently developed continuous flutter constraint is a parametric constraint and maybe replaced by the equivalent minimum value constraint. An example problem solved using the continuous flutter constraint demonstrated that some problems cannot be solved using more conventional flutter constraints.

Appendix

Calculation of Derivatives for the Continuous Flutter Constraint

If the second-order approximation [Eq. (15)] is to be used the derivatives of $G_f(k)$ are needed. The derivatives of $G_f(k)$ are obtained easily from the derivatives of g and V (see also Ref. 10). To obtain $(\partial g/\partial k)$ and $(\partial V/\partial k)$ Eq. (18) is differentiated

$$\left[\frac{\partial C}{\partial k} - \frac{\partial \Omega_r}{\partial k} I\right]\{q\} + [C - \Omega_r I]\left\{\frac{\partial q}{\partial k}\right\} = 0 \qquad \text{(A1)}$$

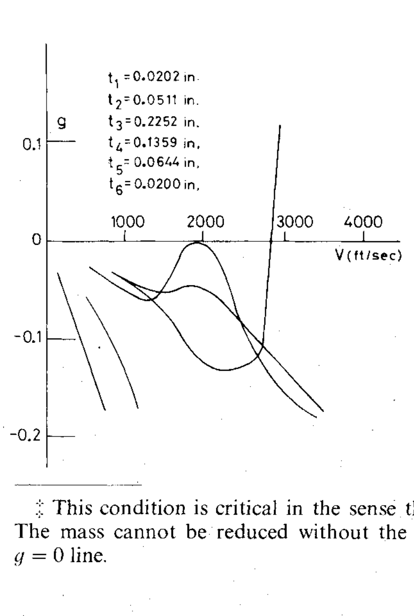

Fig. 5 V-g diagram of final design.

‡ This condition is critical in the sense that it controls the design. The mass cannot be reduced without the hump mode crossing the $g = 0$ line.

or

$$\{q^*\}^T\left[\frac{\partial C}{\partial k}-\frac{\partial \Omega_r}{\partial k}I\right]\{q\}=0 \tag{A2}$$

where $\{q^*\}$ is the adjoint flutter eigenvector which solves the equation

$$\{q^*\}^T[C-\Omega_r I]=0 \tag{A3}$$

From Eq. (A2)

$$\frac{\partial \Omega_r}{\partial k}=\{q^*\}^T\left[\frac{\partial C}{\partial k}\right]\{q\}/\{q^*\}^T\{q\} \tag{A4}$$

After $(\partial\Omega_r/\partial k)$ is calculated from Eq. (A4), $(\partial g/\partial k)$, and $(\partial\omega/\partial k)$ may be calculated by differentiating Eq. (20)

$$\frac{\partial \Omega_r}{\partial k}=-2\left(\frac{\partial \omega}{\partial k}\bigg/\omega\right)\Omega_r+i\frac{\omega_r^2}{\omega^2}\frac{\partial g}{\partial k} \tag{A5}$$

Note that Eq. (A5) is one complex equation for the two real unknowns $(\partial\omega/\partial k)$ and $(\partial g/\partial k)$. Since $k=b_o\omega/V$

$$\partial V/\partial k=-b_o\omega/k^2+[b_o(\partial\omega/\partial k)/k] \tag{A6}$$

One may obtain in a similar manner expressions for the second derivatives (see, for example, Ref. 10), but these involve derivatives of the flutter eigenvectors. It is possibly better to get these second derivatives by finite-differencing the first derivatives.

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